Definition: A type A is contractible, if there is $a : A$, called the center of contraction, such that for all $x : A$, $a = x$.

Definition: A map $f : A \to B$ is an equivalence, if for all $y : B$, its fiber, $\{x : A \mid fx = y\}$, is contractible. We write $A \simeq B$, if there is an equivalence $A \to B$.

Lemma: For each type A, the identity map, $1_A := \lambda_{x:A} x : A \to A$, is an equivalence.

Proof: For each $y : A$, let $\{y\}_A := \{x : A \mid x = y\}$ be its fiber with respect to 1_A and let $\bar{y} := (y, r_A y) : \{y\}_A$. As for all $y : A$, $(y, r_A y) = y$, we may apply Id-induction on y, $x : A$ and $z : (x = y)$ to get that

$$(x, z) = y$$

. Hence, for $y : A$, we may apply Σ -elimination on $u : \{y\}_A$ to get that $u = y$, so that $\{y\}_A$ is contractible. Thus, $1_A : A \to A$ is an equivalence. $\Box$

Corollary: If U is a type universe, then, for $X, Y : U$,

$$(*) X = Y \to X \simeq Y$$

.

Proof: We may apply the lemma to get that for $X : U$, $X \simeq X$. Hence, we may apply Id-induction on $X, Y : U$ to get that $(*)$. $\Box$

Definition: A type universe U is univalent, if for $X, Y : U$, the map $E_{X,Y} : X = Y \to X \simeq Y$ in $(*)$ is an equivalence.